MATH 2460 EXAM 4

NAME Key GRADE ——— OUT OF ~~20~~ 15 PTS

Answer the following questions correctly (***NO** decimal answer!*) for a full credit.

PART I-SERIES AND FUNCTIONS

1. (**2pts**) Given the power series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$, answer the following questions:

   (a) what is the center of the series?

   (b) use the ratio test to find the radius of the series.

   (c) based on the value (if any) of the radius obtained in (b), determine a possible interval of convergence of the series.

   (d) now, check for convergence at the endpoints of the interval obtained in (c) to establish a definitive interval of convergence for the original series.

(a) Center $\boxed{c = 2}$

(b) Let $a_n = \frac{(x-2)^n}{n2^n}$ $\quad a_{n+1} = \frac{(x-2)^n(x-2)}{(n+1)2^n \cdot 2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)\,n}{(n+1)\,2}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= \frac{|x-2|}{2} < 1 \longrightarrow |x-2| < 2$$

So radius of convergence $\boxed{R = 2}$

(c) Possible interval of convergence $(0, 4)$

(d) When $x = 0$ $\sum \frac{(-1)^{n+1}(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$ diverges

b/c $\sum \frac{(-1)^{2n+1}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series

When $x = 4$, $\sum \frac{(-1)^{n+1}}{n}$ converges (conditionally) b/c it is an alternative harmonic series. Hence $\boxed{x \in (0, 4]}$ OR $\boxed{0 < x \leq 4}$

2. (**2pts**) Find a power series for the function, $f(x) = \dfrac{2}{3x+2}$, centered at -1. (There is no need to determine its convergence!)

$$f(x) = \frac{2}{2+3x} = \frac{2}{-1-(-3)(x+1)} = \frac{-2}{1-\underbrace{3(x+1)}_{r}}$$

$$= \sum (-2)\, 3^n (x+1)^n (-1)^n \quad \text{REMOVE THIS PART} \quad = \sum 2 \text{ REMOVE } (-1)^n (x+1)^n (3)^n$$

or

$$= \sum \text{REMOVE } (-1)^{2n+1}\, 2 \cdot \text{REMOVE } 3^n (x+1)^n$$

PART II-PARAMETRIC EQUATIONS

3. (**2pts**) Consider the parametric equations: $x = \sin(t)$ and $y = 2\cos(t)$. Find:

   (a) dy/dx and d^2y/dx^2

   (b) the slope and concavity (if possible) when $t = \frac{\pi}{3}$. (If an answer does not exist, write DNE.)

$$\frac{dx}{dt} = +\cos(t) \qquad \frac{dy}{dt} = -2\sin t$$

$$\frac{dy}{dx} = \frac{-2\sin t}{\cos t} = -2\tan t \longrightarrow \frac{d^2y}{dx^2} = \frac{-2\sec^2 t}{\cos t} = -2\sec^3 t$$

at $t = \frac{\pi}{3}$, $\quad -2\sec^3\left(\frac{\pi}{3}\right) = -2\left(\frac{1}{\cos\frac{\pi}{3}}\right)^3 = -2(8) = \boxed{-16}$

Concave downward!

$t = \frac{\pi}{3}$, $\quad \boxed{-2\sqrt{3}}$

4. (**2pts**) Find the area of the surface generated by revolving the parametric curve $c(t) = (7t, 9t)$ about:

$0 \le t \le 7$

(a) x-axis

(b) y-axis

(a) $A = 2\pi \int_0^7 9t\sqrt{49+81}\, dt$ since $\frac{dx}{dt} = 7$ $\frac{dy}{dt} = 9$

$= 18\sqrt{130}\pi \left[\frac{t^2}{2}\right]_0^7 = 441\sqrt{130}\pi$ unit^2

(b) $A = 2\pi \int_0^7 7t\sqrt{49+81}\, dt$

$= 14\sqrt{130}\pi \left[\frac{t^2}{2}\right]_0^7$

$= 343\sqrt{130}\pi$ unit^2

5. (**1pt**) Find the polar coordinates of the point $(\frac{4}{\sqrt{3}}, 4)$ with $r > 0$ and $0 \le \theta < 2\pi$

$r = \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 + (4)^2} = \frac{8}{\sqrt{3}}$ $\theta = \tan^{-1}\left(\frac{4}{4/\sqrt{3}}\right)$

$= \frac{\pi}{3}$

So $(r, \theta) = \left(\frac{8}{\sqrt{3}}, \frac{\pi}{3}\right)$

6. (6pts) Find the polar coordinates of the points on the graph of $r = 1 - \cos(\theta)$ where the tangent line is horizontal, with $r \geq 0$ and $0 \leq \theta < 2\pi$.

$$x = r\cos\theta = \cos\theta - \cos^2\theta$$
$$y = r\sin\theta = \sin\theta - \sin\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$
$$= \frac{-2\cos^2\theta + \cos\theta + 1}{-\sin\theta\,(1 + 2\cos\theta)}$$

$$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta\sin\theta$$
$$\frac{dy}{d\theta} = -2\cos^2\theta + \cos\theta + 1$$

So $\frac{dy}{dx} = 0 \longrightarrow$ Numerator: $(\cos\theta - 1)(\cos\theta + \frac{1}{2}) = 0$ ⓐ

Denominator: $-\sin\theta\,(1 + 2\cos\theta) = 0$ ⓑ

ⓐ $\cos\theta = 1 \longrightarrow \theta = 0$

$\cos\theta = -\frac{1}{2} \longrightarrow \bar{\theta} = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \longrightarrow$

$\pi - \bar{\theta}$ $\quad \bar{\theta} = \frac{\pi}{3}$ $\quad \pi + \bar{\theta}$

$\theta = \frac{2\pi}{3}$; $\frac{4\pi}{3}$

So $(r, \theta) = \boxed{\left(\frac{3}{2}, \frac{2\pi}{3}\right)}$; $\boxed{\left(\frac{3}{2}, \frac{4\pi}{3}\right)}$

ⓑ $\sin\theta = 0 \longrightarrow \theta = \pi \longrightarrow (r, \theta) = \boxed{(2, \pi)}$

$\cos\theta = -\frac{1}{2}$ (Done!)